

On the Riccati Equations for the Scattering  
Matrices in Two Dimensions

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Research Report YALEU/DCS/RR-1081  
July 20, 1995

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19950803 031

This is the first of a series of papers addressing the solution of the inverse scattering problem for the Helmholtz equation in two dimensions. Here, we derive a system of differential equations for the scattering matrices which

1. Directly govern the whole behavior of the scattering problem,
2. Can be easily implemented numerically with any prescribed accuracy.

In the subsequent papers, we will use this apparatus to design stable inversion algorithms for the acoustical inverse scattering problem. Specifically, in the second paper, we will present a scheme based on the trace formula which is a direct extension to the one employed in [16]. The algorithm is quite satisfactory analytically, but requires excessive amounts of CPU time. Finally, in the third paper, we will present a radically accelerated version of the algorithm.

# **On the Riccati Equations for the Scattering Matrices in Two Dimensions**

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The authors were supported in part by the ONR under the grants ONR N00014-89-J-1527 and N00014-93-1-0114.

Approved for public release: distribution is unlimited.

**Keywords:** *Helmholtz Equation, Inverse Scattering, Scattering Matrix, Riccati Equation.*

# 1 Introduction

During the last several decades, the inverse scattering problems for the Helmholtz equation have enjoyed a remarkable degree of popularity, both in pure and applied contexts (see, e.g., [2], [3]). A number of algorithms have been proposed for the numerical treatment of these problems, in such environments as medical diagnostics, non-destructive testing, anti-submarine warfare, oil exploration, etc. In the design of such a scheme, three major problems have to be overcome.

1. The problem is highly non-linear, even in its purely mathematical form. In the one-dimensional case, the problem can be reduced to a linear one, but the procedure is not stable numerically.
2. Once a mathematically valid inversion scheme is constructed, it may or may not be stable numerically. In fact, no numerically robust schemes seem to exist at this time, except in one dimension.
3. The cost of applying the scheme on the computer tends to be extremely high, except in the one-dimensional case.

The existing attempts to solve inverse scattering problems for the Helmholtz equation can be roughly subdivided into four groups.

1. Linearized inversion schemes, attempting to approximate the inverse scattering problem by the problem of inverting an appropriately chosen linear operator (see, for example, [3]).
2. Methods based on non-linear optimization techniques, attempting to recover the parameters of the problem iteratively, by solving a sequence of forward scattering problems (see, for example, [4], [5], [6]).
3. Gel'fand-Levitan and related techniques, converting the Helmholtz equation into the Schrödinger equation, the inverse problem for the latter being reducible to the solution of a linear Volterra integral equation (see, for example, [2], [7]).
4. Techniques based on the trace formulae, connecting the high-frequency behavior of the solutions of the Helmholtz equation with the local values of the parameters to be recovered (see, for example, [10], [11], [12]).

From the mathematical viewpoint, the one-dimensional problem was satisfactorily solved in the early fifties (see [7], [8]). However, procedures of the type described in [7] and [8] do not lead to stable numerical algorithms. Existing stable and efficient schemes in one dimension are based on the trace formulae. They consist of constructing a Riccati equation for some function of the solution of the Helmholtz equation (such as the impedance or the scattering coefficient), and combining it with a trace formula. The result is a system of ordinary differ-

ential equations that can be solved numerically, and a proper choice of the trace formula ensures stability and rapid convergence of the process (see, for example, [10], [11], and [16]).

Attempts to generalize this approach to higher dimensions have not lead to effective numerical procedures; however, a collection of powerful mathematical apparatus has been developed (see [9] and [10] for quantum scattering in one dimension; [13], [14] and [17] for electrical impedance scattering; and [15] for Schrödinger scattering in three dimensions). One of the more promising tools developed to-date is the concept of the scattering matrix (or of the Dirichlet-to-Neumann map) and the differential equations governing it (see [13], [14], [17]).

The present work is the beginning of a series of papers addressing the solution of the inverse scattering problem for the Helmholtz equation in two dimensions. In this paper, our goal is to derive a system of differential equations for the scattering matrices which

1. Directly governs the whole behavior of the scattering problem,
2. Can be easily implemented numerically with any prescribed precision.

In the subsequent papers, we will use this apparatus to design the inversion algorithms. Specifically, in the second paper, we will present an inverse scattering scheme based on the trace formula which is a direct extension to the one employed in [16]. The algorithm is quite satisfactory analytically, but requires excessive amounts of CPU time. Finally, in the third paper to be published, we will present a much accelerated version of the algorithm.

The principal results of the paper are the Riccati equations for the scattering matrices in cylindrical coordinates. The paper is organized as follows. In Section 2, we summarize the relevant properties of the Helmholtz equation in two dimensions, and introduce the scattering matrices. In Section 3, we derive the differential equations which the scattering matrices satisfy.

## 2 Mathematical Preliminaries

In this section, we will discuss the Helmholtz equation and its associated scattering problems. First in Section 2.1, we introduce common and special usages of notation in this paper. In Section 2.2, properties of the Bessel functions are presented. Section 2.3 is then devoted to the scattering problems for the Helmholtz equation. Finally in Sections 2.5 and 2.6, we define scattering matrices corresponding to three special scattering problems.

## 2.1 Notation

We will denote by  $C^+$  the set of all complex numbers with nonnegative imaginary part, and by  $S^1 \subset C$  the unit circle in the complex plane defined by the formula

$$|z| = 1; \quad (1)$$

we will assume that  $S^1$  is parameterized by its arc length. We will define  $\ell$  as the linear space of all two-sided sequences of complex numbers

$$\ell = \{ \{ \xi_m \}, m = 0, \pm 1, \pm 2, \dots \}. \quad (2)$$

In agreement with standard practice, we will denote by  $\ell^2$  the subspace of  $\ell$  consisting of all sequences  $\xi$  such that

$$\sum_{m=-\infty}^{\infty} |\xi_m|^2 < \infty, \quad (3)$$

and by  $\ell^\infty$  the subspace of  $\ell$  consisting of all sequences  $\xi$  such that

$$\sup_m |\xi_m| < \infty. \quad (4)$$

Let  $F : L^2(S^1) \mapsto \ell^2$  denote the Fourier transform converting a square integrable function on the circle  $S^1$  into its Fourier series, so that the expression

$$f(\theta) = \sum_{m=-\infty}^{+\infty} \xi_m e^{im\theta} \quad (5)$$

can be written in the matrix form

$$f = F^{-1}\xi. \quad (6)$$

When it is necessary to explicitly show the dependence of  $F^{-1}\xi$  on the variable  $\theta$ , the subscript  $\theta$  will be suffixed to the linear mapping  $F^{-1}$ ; therefore the expression (6) can be rewritten as

$$f(\theta) = F_\theta^{-1}\xi. \quad (7)$$

In agreement with standard practice, we denote by  $J_m$  the Bessel function of the first kind of order  $m$ , and by  $H_m$  the Hankel function of the first kind of order  $m$ . We will denote by  $J_z$ ,  $H_z$  the infinite diagonal matrices

$$J_z = \text{diag}\{\dots, J_{-1}(z), J_0(z), J_1(z), \dots\}, \quad (8)$$

$$H_z = \text{diag}\{\dots, H_{-1}(z), H_0(z), H_1(z), \dots\}. \quad (9)$$

Given  $R > 0$ , we will denote by  $D(R)$  the disk

$$D(R) = \{ (r, \theta) \mid r \leq R \}, \quad (10)$$

by  $E(R)$  the exterior of  $D(R)$

$$E(R) = \{ (r, \theta) \mid r \geq R \}, \quad (11)$$

and by  $A(R, h)$  for  $h > 0$  the annulus

$$A(R, h) = \{ (r, \theta) \mid R \leq r \leq R + h \}. \quad (12)$$

Further, for an arbitrary  $x \in R^2$ , we will denote by  $D(x, R)$  the disk of radius  $R$  centered at  $x$ . For a continuous function  $q \in C(D(R))$ , and a real number  $r \leq R$ , we define the mapping  $Q_r : C(S^1) \mapsto C(S^1)$  by the formula

$$(Q_r \cdot f)(\theta) = q(r, \theta) \cdot f(\theta). \quad (13)$$

Given a function  $q : R^2 \mapsto R^1$ , and a set  $\mathcal{A} \subset R^2$ , we will define the function  $q_{\mathcal{A}}$ , the restriction of  $q$  on  $\mathcal{A}$ , via the formula

$$q_{\mathcal{A}}(x) = q(x) \cdot \chi(\mathcal{A}), \quad (14)$$

with  $\chi(\mathcal{A})$  the characteristic function of  $\mathcal{A}$ .

For an arbitrary  $z \in C$ , we will denote by  $X_z$  the linear space of all two-sided complex sequences  $\{\beta_m\}$  such that for some  $c > 0$ ,

$$|\beta_m| \cdot \sqrt{|m|} \cdot \left( \frac{2|m|}{e \cdot z} \right)^{|m|} < c \quad (15)$$

for all integer  $m$ . We will denote by  $Y_z$  the linear space of all two-sided complex sequences  $\{\alpha_m\}$  such that for some  $c > 0$ ,

$$|\alpha_m| \cdot \sqrt{|m|} \cdot \left( \frac{e \cdot z}{2|m|} \right)^{|m|} < c \quad (16)$$

for all integer  $m$ .

## 2.2 Several Classical Lemmas

In this subsection, we summarize several classical results describing the behavior of certain special solutions of the Helmholtz equation. The following lemma can be found, for example, in [21].

**Lemma 2.1** *Let  $m$  be an integer. Suppose that  $J_m$  is the Bessel function of order  $m$ , and  $Y_m$  is Neumann function of order  $m$ . Suppose further that  $H_m$  is the first kind Hankel function of order  $m$  defined by the formula*

$$H_m = J_m + i \cdot Y_m. \quad (17)$$

*Then*

$$(I) \ J_m = (-1)^m J_{-m}, \text{ and } H_m = (-1)^m H_{-m}.$$

$$(II) \text{ For any } z \in C^+,$$

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}, \quad (18)$$

$$Y_0(z) = \frac{\pi}{2} \left\{ \ln \left(\frac{z}{2}\right) + \gamma \right\} J_0(z) - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \sum_{m=1}^n \frac{1}{m}. \quad (19)$$

*with  $\gamma = 0.5772 \dots$  the Euler constant.*

$$(III) \text{ For any } z \in C^+,$$

$$\lim_{m \rightarrow \infty} J_m(z) \cdot \sqrt{2\pi m} \cdot \left(\frac{2m}{e \cdot z}\right)^m = 1, \quad (20)$$

*and*

$$\lim_{m \rightarrow \infty} H_m(z) \cdot \sqrt{\frac{\pi m}{2}} \cdot \left(\frac{e \cdot z}{2m}\right)^m = i. \quad (21)$$

*In other words, for a fixed  $z \in C^+$  as  $m$  increases,  $J_m(z)$  decays as*

$$J_m(z) \sim \frac{1}{\sqrt{2\pi m}} \cdot \left(\frac{e \cdot z}{2m}\right)^m, \quad (22)$$

*and  $H_m(z)$  grows as*

$$H_m(z) \sim -i \cdot \sqrt{\frac{2}{\pi m}} \cdot \left(\frac{2m}{e \cdot z}\right)^m. \quad (23)$$

$$(IV) \text{ For a fixed } m \text{ and large } z \in C^+,$$

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos \left( z - m \frac{\pi}{2} - \frac{\pi}{4} \right) + O(z^{-1}) \right\}, \quad (24)$$

$$H_m(z) = \sqrt{\frac{2}{\pi z}} \left\{ \exp \left[ i \cdot \left( z - m \frac{\pi}{2} - \frac{\pi}{4} \right) \right] + O(z^{-1}) \right\}. \quad (25)$$

*More specifically, for  $z \in C^+$  and  $|z| \geq 2$ ,*

$$H_0(z) = \sqrt{\frac{2}{\pi z}} \left\{ e^{i(z-\pi/4)} + \epsilon(z) \right\} \quad (26)$$

with

$$|\epsilon(z)| \leq \frac{1}{8 \cdot |z|}. \quad (27)$$

The following lemma provides an upper bound for the Hankel function  $H_0$ .

**Lemma 2.2** *Suppose that  $z \in C^+$  is a complex number in the upper half of the complex plane. Then*

$$|H_0(z)| < 2 \quad (28)$$

for all  $|z| \geq 2$ , and

$$|H_0(z)| < 2 + |\ln(|z|/2)| \quad (29)$$

for all  $|z| < 2$ .

**Proof.** For  $|z| \geq 2$ , formula (28) follows directly from (26). For  $|z| < 2$ , formula (29) follows directly from the combination of (17), (18) and (19).  $\square$

For a domain  $\Omega \subset R^2$ , a point  $x \in R^2$ , and a positive number  $k$ , we will denote by  $|\Omega|$  the area of  $\Omega$ , and by  $P(x, \Omega)$  the real number defined by the formula

$$P(x, \Omega) = \int_{\Omega \cap D(x, 1/k)} |\ln(k\|x - \xi\|)| d\xi. \quad (30)$$

The following technical lemmas provide estimates for  $P(x, \Omega)$ .

**Lemma 2.3** *For arbitrary domain  $\Omega \subset R^2$ , point  $x \in R^2$ , and positive number  $k \in R^1$ ,*

$$P(x, \Omega) \leq P(x, D(x, \rho)) \quad (31)$$

provided that

$$\rho = \sqrt{\frac{|\Omega|}{\pi}} \leq \frac{1}{k}. \quad (32)$$

**Proof.** It obviously follow from (32) that

1. The domain  $\Omega$  and the disk  $D(x, \rho)$  have the same area,
2.  $D(x, \rho)$  is contained in  $D(x, 1/k)$

Assuming that  $\Omega$  does not coincide with  $D(x, \rho)$ , that is, assuming that the two domains

$$A = \Omega \setminus D(x, \rho), \quad (33)$$

$$B = D(x, \rho) \setminus \Omega \quad (34)$$



have non-zero areas  $|A| = |B|$ , we intend to show that

$$P(x, \Omega) < P(x, D\left(x, \sqrt{\frac{|\Omega|}{\pi}}\right)). \quad (35)$$

Without loss of generality, we may assume that  $\Omega$  is contained in  $D(x, 1/k)$ . It follows from (32) that for any  $\xi_1 \in A \subset D(x, 1/k)$  and  $\xi_2 \in B \subset D(x, 1/k)$

$$k\|x - \xi_2\| \leq k \cdot \rho < k\|x - \xi_1\| \leq 1, \quad (36)$$

we have

$$\|x - \xi_2\| < \|x - \xi_1\|. \quad (37)$$

Therefore,

$$|\ln(k\|x - \xi_1\|)| < |\ln(k\|x - \xi_2\|)| \quad (38)$$

which establishes (31).  $\square$

**Lemma 2.4** *For arbitrary domain  $\Omega \subset R^2$  and point  $x \in R^2$ ,*

$$P(x, \Omega) \leq \frac{|\Omega|}{2} \left\{ 1 + \left| \ln \left( k \sqrt{\frac{|\Omega|}{\pi}} \right) \right| \right\}. \quad (39)$$

**Proof.** We first consider the case when the area of  $\Omega$  is no less than that of the disk  $D(x, 1/k)$ , that is,

$$|\Omega| \geq \frac{\pi}{k^2}. \quad (40)$$

Under this assumption, it immediately follows from (30) that

$$\begin{aligned} P(x, \Omega) &\leq \int_{D(x, 1/k)} |\ln(k\|x - \xi\|/2)| d\xi \\ &= - \int_0^{2\pi} \int_0^{1/k} \ln(kr) r dr d\theta \\ &= \frac{\pi}{2k^2} \leq \frac{|\Omega|}{2}, \end{aligned} \quad (41)$$

from which (39) follows. Now, we consider the case when the area of  $\Omega$  is less than that of the disk  $D(x, 1/k)$ , that is,

$$|\Omega| < \frac{\pi}{k^2}. \quad (42)$$

Let

$$\rho = \sqrt{\frac{|\Omega|}{\pi}}. \quad (43)$$

According to Lemma 2.3,

$$\begin{aligned}
P(x, \Omega) &\leq P(x, D(x, \rho)) = \int_{D(x, \rho)} |\ln(k\|x - \xi\|)| \, d\xi \\
&= - \int_0^{2\pi} \int_0^\rho \ln(kr) r \, dr \, d\theta \\
&= \frac{|\Omega|}{2} \left\{ 1 + 2 \cdot \left| \ln \left( |k| \sqrt{\frac{|\Omega|}{\pi}} \right) \right| \right\}.
\end{aligned} \tag{44}$$

□

**Definition 2.5** *In agreement with standard usage, we will refer to the function  $f : R^2 \mapsto C$  in the Helmholtz equation*

$$\Delta\phi + k^2\phi = f \tag{45}$$

*as the source, and to the solution  $\phi : R^2 \mapsto C$  as the field generated by the source.*

The following two lemmas can be found, for example, in [1].

**Lemma 2.6** *Suppose that  $k \in C^+$  is a complex number. Then the Green's function for the homogeneous Helmholtz equation*

$$\Delta\phi + k^2\phi = 0 \tag{46}$$

*subject to the Sommerfeld radiation condition*

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial \phi}{\partial r} - ik\phi \right) = 0 \tag{47}$$

*is given by the formula*

$$G_k(x, \xi) = -\frac{i}{4} H_0(k\|x - \xi\|), \tag{48}$$

*where  $x = (r, \theta)$  and  $\xi = (r', \theta')$  are arbitrary distinct points in  $R^2$ .*

**Lemma 2.7** (*Graf's Addition Formula*) *Suppose that the positive numbers  $u, v, w, \alpha$  are such that  $u > v$ , and that*

$$w^2 = u^2 + v^2 - 2uv \cos(\alpha). \tag{49}$$

*Then*

$$H_0(w) = \sum_{m=-\infty}^{\infty} H_m(u) J_m(v) e^{im\alpha}. \tag{50}$$

The following lemma is an immediate consequence of Lemma 2.7.

**Lemma 2.8** *Suppose that  $x, \xi$  are two points in  $R^2$  such that  $\|x\| > \|\xi\|$ . Suppose further that  $(r, \theta)$  and  $(r', \theta')$  are the polar coordinates of the vectors  $x, \xi$ , respectively. Then*

$$H_0(k\|x - \xi\|) = \sum_{m=-\infty}^{\infty} H_m(kr)J_m(kr')e^{im(\theta-\theta')}. \quad (51)$$

The following lemma is a direct result of Lemmas 2.6 and 2.2.

**Lemma 2.9** *Suppose that  $k \in C^+$  is non-zero, and that  $\Omega \subset R^2$  is a domain of area  $\mu$ . Then*

$$\int_{\Omega} |G_k(x, \xi)| d\xi < \frac{\frac{5}{2} + \left| \ln \left( \frac{|k|}{2} \cdot \sqrt{\frac{\mu}{\pi}} \right) \right|}{4} \mu \quad (52)$$

for all  $x \in R^2$ .

**Proof.** Introducing the notation

$$I(x) = \int_{\Omega} |G_k(x, \xi)| d\xi \quad (53)$$

and using (48), we have

$$I(x) = \frac{1}{4} \left( \int_{\Omega \cap \{|z| > 1\}} |H_0(z)| d\xi + \int_{\Omega \cap \{|z| < 1\}} |H_0(z)| d\xi \right), \quad (54)$$

with  $z = k\|x - \xi\|/2$ . Combining (28) and (29) with (54), we immediately obtain the estimate

$$I(x) < \frac{1}{4} \left( 2\mu + \int_{\Omega \cap \{|z| < 1\}} |\ln(|z|)| d\xi \right) \quad (55)$$

for all  $x \in R^2$ . Now the lemma follows immediately from (55) and Lemma 2.4.  $\square$

**Remark 2.10** *Denoting by  $G_{k,\Omega}$  the linear operator  $C(\Omega) \mapsto C(\Omega)$  defined by the the formula*

$$(G_{k,\Omega} \cdot \psi)(x) = \int_{\Omega} G_k(x, \xi) \psi(\xi) d\xi, \quad (56)$$

for all  $\psi \in C(\Omega)$ , we can rewrite (52) in the form

$$\|G_{k,\Omega}\|_{\infty} = \frac{\frac{5}{2} + \left| \ln \left( \frac{|k|}{2} \cdot \sqrt{\frac{\mu}{\pi}} \right) \right|}{4} \mu \quad (57)$$

The following lemma provides a sharper upper bound for  $\|G_{k,\Omega}\|_\infty$  when  $\Omega$  is the annulus  $A(R, h)$  (see (12)). Its proof is nearly identical to that of Lemma 2.9 and is omitted.

**Lemma 2.11** *Suppose that  $R \geq 0$ ,  $h > 0$  are two real numbers, and that  $k \in C^+$  is non-zero. Then for all  $x \in R^2$*

$$\int_{A(R,h)} |G_k(x, \xi)| d\xi < \mu \quad (58)$$

where  $\mu = h \cdot \pi \cdot (2R + h)$  is the area of the annulus  $A(R, h)$ .

**Remark 2.12** *The preceding two lemmas show that there are two types of estimates (see formulae (52) and (58)) on  $\|G_{k,\Omega}\|_\infty$ , depending on the shape of the domain. When the domain is a disk, the estimate is of the form*

$$\|G_{k,\Omega}\|_\infty < c \cdot |\ln(\mu)| \cdot \mu, \quad (59)$$

whereas when the domain is an annulus, the estimate is of the form

$$\|G_{k,\Omega}\|_\infty < c \cdot \mu. \quad (60)$$

**Definition 2.13** *A function  $\phi$  is said to be a radiation field in a bounded domain  $\Omega$  if and only if  $\phi$  is a solution of the homogeneous Helmholtz equation (46) in  $\Omega$ ; a function  $\phi$  is said to be a radiation field outside a bounded domain  $\Omega$  if and only if  $\phi$  is a solution of the homogeneous Helmholtz equation (46) subject to the Sommerfeld radiation condition (47).*

**Lemma 2.14** *Suppose that  $k$  is an arbitrary complex number,  $R$  is a positive real number, and  $\phi : D(R) \mapsto C$  is a radiation field in  $D(R)$ . Then there exists a sequence of complex numbers  $\{\alpha_j\}$ ,  $j = 0, \pm 1, \pm 2, \dots$ , such that*

$$\phi(r, \theta) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(kr) e^{im\theta} = F_\theta^{-1} J_{kr} \alpha, \quad (61)$$

for all  $r < R$  and  $0 \leq \theta \leq 2\pi$ .

The following two lemmas are widely known. They can be found, for example, in [1].

**Lemma 2.15** *Suppose that  $k$  is an arbitrary complex number,  $R$  is a positive real number, and  $\psi : R^2 \setminus D(R) \mapsto C$  is a radiation field outside  $D(R)$ . Then there exists a sequence of complex numbers  $\{\alpha_j\}$ ,  $j = 0, \pm 1, \pm 2, \dots$ , such that*

$$\psi(r, \theta) = \sum_{m=-\infty}^{\infty} \beta_m H_m(kr) e^{im\theta} = F_\theta^{-1} H_{kr} \beta, \quad (62)$$

for all  $r < R$  and  $0 \leq \theta \leq 2\pi$ .

**Remark 2.16** *It immediately follows from Lemma 2.1 that the series (61) converges inside  $D(R)$  if and only if  $\alpha \in Y_{kR}$ , and the series (62) converges outside  $D(R)$  if and only if  $\beta \in X_{kR}$ .*

## 2.3 Scattering Problems

The subject of this paper is the Helmholtz equation

$$\Delta\phi + k^2(1 + q)\phi = 0. \quad (63)$$

In (63), we assume that  $k$  is a complex number such that  $\text{Im}(k) \geq 0$ , and  $q : R^2 \mapsto R^1$  is a smooth function; we will be referring to the function  $q$  as the scatterer. We further assume that the support of  $q$  is a bounded domain  $\Omega$ , and that  $q(x) > -1$  for all  $x \in R^2$ . We will be considering solutions  $\phi : \Omega \mapsto C$  of Equation (63) of the form

$$\phi = \phi_0 + \psi, \quad (64)$$

with  $\phi_0$  a radiation field in  $\Omega$ , and  $\psi : R^2 \mapsto C$  a radiation field outside  $\Omega$ . We will be referring to  $\phi$  as the total field, to  $\phi_0$  as the incoming field, and to  $\psi$  as the scattered field. Furthermore, we will be referring to the determination of the scattered field from a given incoming field as the (forward) scattering problem. It is easy to verify that  $\psi(x)$  satisfies the equation

$$\Delta\psi + k^2\psi = -k^2q(\phi_0 + \psi) \quad (65)$$

for all  $x \in R^2$ .

**Remark 2.17** *It is well-known (see, for example, [19], [18], [20]) that the forward scattering problem is well-posed. More specifically, the problem can be reformulated as the so-called Lippmann-Schwinger equation*

$$\psi(x) + k^2 \int_{\Omega} G_k(x, \xi) q(\xi) \psi(\xi) d\xi = -k^2 \int_{\Omega} G_k(x, \xi) q(\xi) \phi_0(\xi) d\xi. \quad (66)$$

*for the scattered field, or as the Lippmann-Schwinger equation*

$$\phi(x) + k^2 \int_{\Omega} G_k(x, \xi) q(\xi) \phi(\xi) d\xi = \phi_0(x). \quad (67)$$

*for the total field, for all  $x \in R^2$ ; either of the second kind integral operators (66), (67) is invertible, and the maximum norm of the inverse operator is bounded. That is to say, for  $k \in C^+$  and  $q \in C(R^2)$  having the compact support  $\Omega$ , an incoming field  $\phi_0$  determines uniquely a scattered field  $\psi$ , and  $\psi$  depends continuously on  $\phi_0$  in maximum norm.*

Defining the linear operator  $G_k^q : C(\Omega) \mapsto C(R^2)$  by the formula

$$(G_k^q \cdot \psi)(x) = k^2 \int_{\Omega} G_k(x, \xi) q(\xi) \psi(\xi) d\xi, \quad (68)$$

we can rewrite (66) in the form

$$\psi + G_k^q \psi = g, \quad (69)$$

with  $g \in C(R^2)$  defined by the formula

$$g = G_k^q \phi_0. \quad (70)$$

When

$$\|G_k^q\|_{\infty} < 1, \quad (71)$$

the equation (69) can be solved via the fixed-point iteration

$$\psi_0 = 0, \quad (72)$$

$$\psi_{m+1} = g - G_k^q \psi_m. \quad (73)$$

It follows from Lemma 2.9 that the condition (71) is met whenever the area of the domain  $\Omega$  is sufficiently small. The above discussion is formalized in the following lemma whose simple proof we omit.

**Lemma 2.18** *Suppose that  $k \in C^+$  is a complex number,  $q \in C(R^2)$ , and  $\Omega \subset R^2$  is a domain of area  $\mu$ . Suppose further that*

$$\delta = \frac{\frac{5}{2} + \left| \ln \left( |k| \cdot \sqrt{\frac{\mu}{\pi}} \right) \right|}{4} |k|^2 \cdot \|q\|_{\infty} \cdot \mu. \quad (74)$$

*Finally, suppose that  $\mu$  is sufficiently small so that*

$$\delta < 1. \quad (75)$$

*Then the fixed-point iteration (72), (73) converges to the solution of Lippmann-Schwinger equation (69). Moreover,*

$$|\psi(x) - \psi_m(x)| \leq \delta^{m+1} \quad (76)$$

*for all integer  $m \geq 0$ , and  $x \in R^2$ .*

When  $\Omega$  is an annulus  $A(R, h)$  with  $h$  small, Lemma 2.18 assumes the form provided by the following lemma.

**Lemma 2.19** *Suppose that  $R \geq 0$  and  $h > 0$  are two real numbers. Suppose further that  $k \in C^+$  is a complex number,  $q \in C(R^2)$ , and  $\Omega = A(R, h)$  is an annulus of width  $h$ . Finally, suppose that  $h$  is less than 1, and that*

$$h < \frac{1}{\pi \cdot \|k\|^2 \cdot \|q\|_\infty (2R + 1)}, \quad (77)$$

*Then the fixed-point iteration (72), (73) converges to the solution of Lippmann-Schwinger equation (69). Moreover,*

$$|\psi(x) - \psi_m(x)| \leq \left( \pi \cdot \|k\|^2 \cdot \|q\|_\infty (2R + 1) \cdot h \right)^{m+1} \quad (78)$$

*for all integer  $m \geq 0$ , and  $x \in R^2$ .*

## 2.4 Scattering in Circular Geometry

In the remainder of this paper, we will be interested in two special cases of the scattering problem: in the first case, the compact support of the scatterer  $q$  is a disk  $D(\rho)$  with some  $\rho > 0$ ; in the second case, the compact support of the scatterer is a domain  $\Omega$  lying outside the disk  $D(\rho)$ , that is,  $\Omega \subset R^2 \setminus D(\rho)$ , see Figure 1.

In this subsection, we construct simple analytical expressions for the incoming and the scattered fields in each of these two cases. Clearly, in the first case, the incoming field  $\phi_0$  is a radiation field inside  $D(\rho)$ , and is therefore generated by sources located outside  $D(\rho)$ ; the scattered field  $\psi$  is a radiation field outside  $D(\rho)$ , and is therefore generated by sources inside  $D(\rho)$ . We will be referring to this scattering problem as the interior scattering problem (see Figure 1:(a)). The following lemma is a direct consequence of Lemmas 2.14, 2.15.

**Lemma 2.20** *Suppose that  $\rho$  is a positive real number, and that the scatterer  $q$  has the compact support  $D(\rho)$ . Then*

*(I) If the function  $\phi_0 : D(\rho) \mapsto C$  is an incoming field to the scatterer  $q$ , then there exists a sequence  $\alpha \in Y_{k\rho}$ , such that for all  $(r, \theta) \in D(\rho)$ ,*

$$\phi(r, \theta) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(kr) e^{im\theta} = F_\theta^{-1} J_{kr} \alpha; \quad (79)$$

*(II) If the function  $\psi : R^2 \mapsto C$  is a scattered field from the scatterer  $q$ , then there exists a sequence  $\beta \in X_{k\rho}$ , such that for all  $(r, \theta) \in R^2 \setminus D(\rho)$ ,*

$$\psi(r, \theta) = \sum_{m=-\infty}^{\infty} \beta_m H_m(kr) e^{im\theta} = F_\theta^{-1} H_{kr} \beta. \quad (80)$$

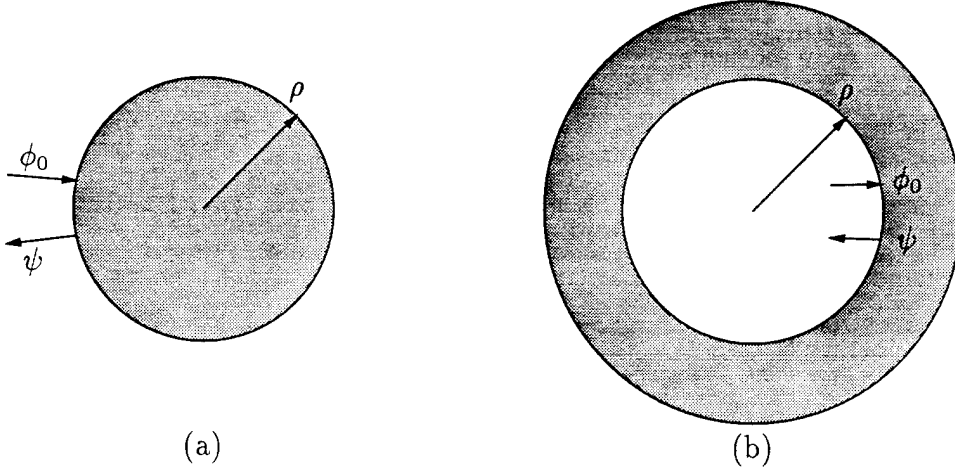


Figure 1: (a) Interior scattering where scatterer occupies the entire disk  $D(\rho)$ . (b) Exterior scattering where the scatterer is located outside the disk  $D(\rho)$ .

In the second case of interest to us, we consider incoming field generated inside the disk  $D(\rho)$ , and therefore the incoming field is a radiation field outside the disk. Since the scattered field  $\psi$  is a radiation field outside  $\Omega$ , it is a radiation field inside  $D(\rho)$ . We will be referring to this scattering problem as the exterior scattering problem (see Figure 1:(b)). The following lemma is a direct consequence of Lemmas 2.14, 2.15.

**Lemma 2.21** *Suppose that  $\rho$  is a positive real number, and that the scatterer  $q$  has a compact support outside  $D(\rho)$ . Then*

(I) *If the function  $\phi_0 : \mathbb{R}^2 \setminus D(\rho) \mapsto \mathbb{C}$  is an incoming field to the scatterer  $q$ , then there exists a sequence  $\beta \in X_{k\rho}$ , such that for all  $(r, \theta) \in \mathbb{R}^2 \setminus D(\rho)$ ,*

$$\phi_0(r, \theta) = \sum_{m=-\infty}^{\infty} \beta_m H_m(kr) e^{im\theta} = F_\theta^{-1} H_{kr} \beta; . \quad (81)$$

(II) *If the function  $\psi : \mathbb{R}^2 \mapsto \mathbb{C}$  is a scattered field of the exterior scattering problem, then there exists a sequence  $\alpha \in Y_{k\rho}$ , such that for all  $(r, \theta) \in D(\rho)$ ,*

$$\psi(r, \theta) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(kr) e^{im\theta} = F_\theta^{-1} J_{kr} \alpha. \quad (82)$$

## 2.5 Scattering Matrices

One of principal analytical tools used in this paper are the scattering matrices; a scattering matrix is the linear mapping converting the incoming field into the scattered field, for a specified scatterer. For technical reasons, it is convenient to



have slightly different definitions of the scattering matrices for the interior and exterior scattering problems; thus, we define the interior and exterior scattering matrices separately.

**Definition 2.22** *Under the conditions of Lemma 2.20, any  $\alpha \in Y_{kR_0}$  defines an incoming field  $\phi_0$  via (79); the resulting scattered field  $\psi$  can be represented in the form (80), with  $\beta \in X_{kR_0}$ . Thus, there exists a linear mapping  $S_k^q : Y_{kR_0} \mapsto X_{kR_0}$  such that*

$$\beta = S_k^q \cdot \alpha, \quad (83)$$

*for all  $\alpha \in Y_{kR_0}$ , and we will refer to  $S_k^q$  as the interior scattering matrix corresponding to the scatterer  $q$  and frequency  $k$ .*

**Remark 2.23** *Obviously, for a fixed  $k \in C^+$ , the scattering matrix contains all the information that can be acquired by any scattering experiments performed outside the scatterer; we will refer to the collection of the scattering matrices*

$$\{ S_k^q \mid k \in R^1 \} \quad (84)$$

*as the complete scattering data.*

**Remark 2.24** *In an actual scattering experiment, measurements are obtained at a finite collection of frequencies  $k_j$ . Therefore, in a more realistic formulation of the inverse scattering problem, the scatterer  $q$  is to be determined from measurements of the scattering matrices at finite number of frequencies:*

$$\{ S_{k_j}^q \mid j = 1, 2, \dots, N \}. \quad (85)$$

In the remainder of this paper, we will assume that our scatterer  $q$  has compact support  $D(R_0)$  for some positive number  $R_0$ , so that

$$q(r, \theta) = 0 \quad (86)$$

for all  $r \geq R_0$ . Furthermore, for a positive number  $R$ , we will denote by  $S_{R,k}^-$  the interior scattering matrix corresponding to the scatterer  $q_{D(R)}$  (see (14)) and frequency  $k$ ; in other words,

$$S_{R,k}^- = S_k^{q_{D(R)}}. \quad (87)$$

We will refer to the function  $q_{D(R)}$  as the truncated scatterer. Obviously, at  $R = 0$ , the truncated scatterer is zero; it therefore generates no scattered field; in other words,

$$S_{0,k}^- = 0. \quad (88)$$

Likewise, since  $q_{D(R)} \equiv q$  for any  $R \geq R_0$ , we have

$$S_{R,k}^- = S_k^q \quad (89)$$

for all  $R \geq R_0$ .

Closely related to operators  $S_{R,k}^-$  are the so-called exterior scattering matrices associated with the scatterer  $q_{E(R)}$  (see (11),(14)).

**Definition 2.25** *Under the conditions of Lemma 2.21, any  $\beta \in X_{kR}$  defines an incoming field  $\phi_0$  via (81); the resulting scattered field  $\psi$  can be represented in the form (82), with some  $\alpha \in Y_{kR}$ . Thus, there exists a linear mapping  $S_{R,k}^+ : Y_{kR} \mapsto X_{kR}$ , such that*

$$\alpha = S_{R,k}^+ \cdot \beta \quad (90)$$

for all  $\beta \in X_{kR}$ , and we will refer to  $S_k^+$  as the exterior scattering matrix corresponding to the scatterer  $q_{E(R)}$  and frequency  $k$ .

**Remark 2.26** *We will refer to  $q_{E(R)}$  as the hollowed scatterer. Since the scatterer  $q$  has compact support in  $D(R_0)$ , the hollowed scatterer  $q_{E(R)}$  is zero outside  $D(R_0)$ . Therefore,*

$$S_{R,k}^+ = 0 \quad (91)$$

for all  $R \geq R_0$ .

## 2.6 Scattering from an Annulus

Given a pair of real numbers  $R \geq 0$  and  $h > 0$ , we will refer to a scatterer  $q : R^2 \mapsto R^1$  as an annular scatterer of inner radius  $R$  and width  $h$  if  $q(r, \theta) = 0$  for all  $r < R$  or  $r > R + h$ . Obviously, for any scatterer  $q$ , the scatterer  $q_{A(R,h)}$  (see (12),(14)) is an annular scatterer.

Conceptually, the incoming field to the annular scatterer  $q_{A(R,h)}$  is generated by sources both inside the disk  $D(R)$  and outside the disk  $D(R + h)$  since it is a radiation field inside the annulus  $A(R, h)$ . By the same token, the scattered field from the annular scatterer  $q_{A(R,h)}$  is a radiation field outside the annulus  $A(R, h)$ , and therefore has the form (61) inside the disk  $D(R)$ , and the form (62) outside the disk  $D(R + h)$ . The following two obvious lemmas formalize these facts.

**Lemma 2.27** *Suppose that  $R, h$  are two positive numbers, and that  $A(R, h)$  is the annulus defined via (12). Then  $\phi_0 : A(R, h) \mapsto C$  is an incoming field to  $A(R, h)$  if and only if there exist two functions  $\phi^{(in)} : D(R + h) \mapsto C$ ,  $\psi^{(in)} : R^2 \setminus D(R) \mapsto C$  such that*

$$\phi_0 = \phi^{(in)} + \psi^{(in)}, \quad (92)$$

where  $\phi^{(in)}$  is a radiation field in  $D(R + h)$ ; in other words, there uniquely exists  $\alpha \in Y_{k(R+h)}$ , such that for all  $(r, \theta) \in D(R + h)$ ,

$$\phi^{(in)}(r, \theta) = F_\theta^{-1} J_{kr} \alpha; \quad (93)$$

and where  $\psi^{(\text{in})}$  is a radiation field outside  $D(R)$ ; in other words, there uniquely exists  $\beta \in X_{kR}$ , such that for all  $(r, \theta) \in R^2 \setminus D(R)$ ,

$$\psi^{(\text{in})}(r, \theta) = F_\theta^{-1} H_{kr} \beta. \quad (94)$$

Furthermore, the decomposition (92) is unique.

**Lemma 2.28** *Under the conditions of Lemma 2.27 Suppose that  $\psi : R^2 \mapsto C$  is the scattered field generated by the annular scatterer  $A(R, h)$ . Then there exist two sequences  $\tilde{\alpha} \in Y_{kR}$  and  $\tilde{\beta} \in X_{k(R+h)}$  such that inside the disk  $D(R)$ ,*

$$\psi(r, \theta) = F_\theta^{-1} J_{kr} \tilde{\alpha}, \quad (95)$$

and outside the disk  $D(R + h)$ ,

$$\psi(r, \theta) = F_\theta^{-1} H_{kr} \tilde{\beta}. \quad (96)$$

**Remark 2.29** *Abusing the notation somewhat, we will be referring to the coefficients  $\alpha, \beta$  in (93), (94) as the incoming potential, the coefficients  $\tilde{\alpha}, \tilde{\beta}$  in (95), (96) as the outgoing potential.*

Obviously, given an annular scatterer  $q_{A(R, h)}$ , the combination of Lemmas 2.27 and 2.28 defines a linear mapping  $Y_{k(R+h)} \times X_{kR} \mapsto X_{k(R+h)} \times Y_{kR}$  converting the incoming potential  $(\alpha, \beta) \in Y_{kR} \times X_{kR}$  into the outgoing potential  $(\tilde{\beta}, \tilde{\alpha}) \in X_{kR} \times Y_{kR}$ . Thus, we are led to the following definition.

**Definition 2.30** *For an annular scatterer  $q_{A(R, h)}$ , the scattering matrix  $S_{R, k}^h$  is the mapping  $Y_{k(R+h)} \times X_{kR} \mapsto X_{k(R+h)} \times Y_{kR}$  such that for any incoming potential  $(\alpha, \beta)$ , the outgoing potential  $(\tilde{\beta}, \tilde{\alpha})$  is given by the formula*

$$\begin{bmatrix} \tilde{\beta} \\ \tilde{\alpha} \end{bmatrix} = S_{R, k}^h \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \quad (97)$$

**Remark 2.31** *The scattering matrix  $S_{R, k}^h$  can obviously be partitioned into four submatrices*

$$S_{R, k}^h = \begin{bmatrix} S_{\beta\alpha} & S_{\beta\beta} \\ S_{\alpha\alpha} & S_{\alpha\beta} \end{bmatrix}, \quad (98)$$

so that

$$\tilde{\beta} = S_{\beta\alpha} \alpha + S_{\beta\beta} \beta, \quad (99)$$

$$\tilde{\alpha} = S_{\alpha\alpha} \alpha + S_{\alpha\beta} \beta. \quad (100)$$

## 2.7 Scattering from Two Disjoint Scatterers

When several disjoint scatterers are irradiated by an incoming field, we wish to calculate the scattered field under the condition that we know how to solve the scattering problem associated with each scatterer separately. In other words, if we are given the scattering matrices of individual scatterers, we would like to combine them and obtain the scattering matrix associated with the whole ensemble of the scatterers. In this subsection, we build the necessary analytical machinery. Although the discussion is confined to two disjoint scatterers, all conclusions can easily be extended to the case of multiple scatterers.

### 2.7.1 Lippmann-Schwinger equation for a Part of the Scatterer

Suppose that the scatterer  $q$  consists of two separate scatterers occupying two disjoint domains  $\Omega_1$  and  $\Omega_2$ :

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad (101)$$

$$\Omega_1 \cup \Omega_2 = \Omega. \quad (102)$$

Then the Lippmann-Schwinger equation (67) can be rewritten either as

$$\phi(x) + k^2 \int_{\Omega_1} G_k(x, \xi) q(\xi) \phi(\xi) d\xi = \phi_0(x) - k^2 \int_{\Omega_2} G_k(x, \xi) q(\xi) \phi(\xi) d\xi, \quad (103)$$

or as

$$\phi(x) + k^2 \int_{\Omega_2} G_k(x, \xi) q(\xi) \phi(\xi) d\xi = \phi_0(x) - k^2 \int_{\Omega_1} G_k(x, \xi) q(\xi) \phi(\xi) d\xi. \quad (104)$$

Introducing the functions  $\psi_1 : R^2 \mapsto C$  and  $\psi_2 : R^2 \mapsto C$  via the formulae

$$\psi_1(x) = -k^2 \int_{\Omega_1} G_k(x, \xi) q(\xi) \phi(\xi) d\xi, \quad (105)$$

$$\psi_2(x) = -k^2 \int_{\Omega_2} G_k(x, \xi) q(\xi) \phi(\xi) d\xi, \quad (106)$$

we observe that (103) can be viewed as the Lippmann-Schwinger equation (see (67)) on the domain  $\Omega_1$  for the total field  $\phi$  corresponding to the incoming field

$$\phi_{01} = \phi_0 + \psi_2. \quad (107)$$

Similarly, we observe that (104) can be viewed as the Lippmann-Schwinger equation on the domain  $\Omega_2$  for the total field  $\phi$  corresponding to the incoming field

$$\phi_{02} = \phi_0 + \psi_1. \quad (108)$$

The following obvious lemma formalizes the above discussion.

**Lemma 2.32** Suppose that the scatterer  $q$  consists of two separate scatterers occupying two disjoint domains  $\Omega_1$  and  $\Omega_2$ . Suppose further that  $\phi_0 : \Omega_1 \cup \Omega_2 \mapsto C$  is the incoming field to the scatterer  $q$  and that  $\phi : \Omega_1 \cup \Omega_2 \mapsto C$  is the corresponding total field. Finally, suppose that the four functions  $\psi_j : R^2 \mapsto C$ ,  $\phi_{0j} : \Omega_j \mapsto C$ , for  $j = 1, 2$ , are defined by the formulae (105)–(108). Then

(I) For  $j = 1, 2$ ,  $\psi_j$  is the scattered field corresponding to the incoming field  $\phi_{0j}$  to the scatterer in  $\Omega_j$ .

(II) The incoming field to one of the scatterers is the superposition of the original incoming field  $\phi_0$  and the scattered field from the other scatterer:

$$\phi_{01} = \phi_0 + \psi_2, \quad (109)$$

$$\phi_{02} = \phi_0 + \psi_1. \quad (110)$$

(III) The total field is the superposition of the original incoming field and the scattered fields from the two disjoint scatterers:

$$\phi = \phi_0 + \psi_1 + \psi_2. \quad (111)$$

### 2.7.2 Merging Two Scatterers

In this subsection, we examine the process by which the Lippmann-Schwinger equations for the regions  $\Omega_1$  and  $\Omega_2$  are merged, producing the Lippmann-Schwinger equation for the domain  $\Omega_1 \cup \Omega_2$ . It turns out that when the area of  $\Omega_1$  is small, the result assumes a particularly simple form (see Lemmas 2.34 and 2.37).

Defining the linear operators  $G_j : C(\Omega_j) \mapsto C(R^2)$  by the formula

$$(G_j \cdot \psi)(x) = k^2 \int_{\Omega_j} G_k(x, \xi) q(\xi) \psi(\xi) d\xi, \quad (112)$$

for  $j = 1, 2$ , and using the formula (111), we rewrite the Lippmann-Schwinger equations (103), (104) in the form

$$(I + G_1) \cdot \psi_1 + G_1 \cdot \psi_2 = -G_1 \cdot \phi_0, \quad (113)$$

$$G_2 \cdot \psi_1 + (I + G_2) \cdot \psi_2 = -G_2 \cdot \phi_0. \quad (114)$$

The following lemma is an immediate consequence of Remark 2.17.

**Lemma 2.33** Under the conditions of Lemma 2.32, suppose that the operators  $P_j : C(\Omega_j) \mapsto C(R^2)$ ,  $B_j : C(\Omega_j) \mapsto C(R^2)$  are defined by the formulae

$$P_j = (I + G_j)^{-1}, \quad (115)$$

$$B_j = -(I + G_j)^{-1} \cdot G_j = -P_j \cdot G_j, \quad (116)$$

for  $j = 1, 2$ . Then each of the operators  $P_1$ ,  $P_2$ ,  $B_1$ ,  $B_2$  is bounded in the maximum norm.

**Lemma 2.34** *Under the conditions of Lemma 2.32, suppose that the estimate (60) is valid on  $\Omega_1$ ; that is, there exists a constant  $c$  such that*

$$\|G_{k,\Omega_1}\|_\infty < c \cdot \mu, \quad (117)$$

where  $\mu$  is the area of  $\Omega_1$ . Then, for all sufficiently small  $\mu$ , there exist unique functions  $\psi_1 : R^2 \mapsto C$  and  $\psi_2 : R^2 \mapsto C$  such that (113) and (114) are satisfied. Furthermore,

$$\psi_1(x) = B_1 \cdot (I + B_2)\phi_0 + O(\mu^2), \quad (118)$$

$$\psi_2(x) = B_2 \cdot \{ I + B_1 \cdot (I + B_2) \} \phi_0 + O(\mu^2), \quad (119)$$

$$\psi_1(x) + \psi_2(x) = \{ B_2 + (I + B_2) \cdot B_1 \cdot (I + B_2) \} \phi_0 + O(\mu^2). \quad (120)$$

**Proof.** Rewriting (113) and (114) in the form

$$\psi_1 = B_1(\psi_2 + \phi_0), \quad (121)$$

$$\psi_2 = B_2(\psi_1 + \phi_0), \quad (122)$$

and substituting (122) into (121), we obtain

$$(I - B_1 \cdot B_2)\psi_1 = B_1(I + B_2)\phi_0. \quad (123)$$

It immediately follows from the combination of (117), (112) that there exists a constant  $c_1 > 0$  such that

$$\|G_1\|_\infty < c_1 \cdot \|q\|_\infty \cdot \mu, \quad (124)$$

for all  $\mu > 0$ . Combining (124) with (116), we immediately see that there exist positive constants  $c_2, c_3$  such that

$$\|B_1\| < c_2 \cdot \mu, \quad (125)$$

$$\|B_1 \cdot B_2\| < c_3 \cdot \mu, \quad (126)$$

for all  $\mu > 0$ . Therefore, for all sufficiently small  $\mu$ ,

$$(I - B_1 \cdot B_2)^{-1} = \sum_{m=0}^{\infty} (B_1 \cdot B_2)^m. \quad (127)$$

Combining (127) with (123), we have

$$\psi_1 = \left( \sum_{m=0}^{\infty} (B_1 \cdot B_2)^m \right) B_1(I + B_2)\phi_0. \quad (128)$$

Finally, combining (128) with (125), (126), we obtain

$$\psi_1 = B_1(I + B_2)\phi_0 + O(\mu^2), \quad (129)$$

which proves (118). The substitution of (129) into (122) yields (119).  $\square$

**Remark 2.35** For any  $u \in C(\Omega_j)$ , the function  $v \in C(R^2)$  defined by the formula

$$v = -(I + G_j)^{-1} \cdot G_j \cdot u = B_j \cdot u \quad (130)$$

is obviously a solution of the Lippmann-Schwinger equation

$$(I + G_j) \cdot v = -G_j \cdot u \quad (131)$$

on the domain  $\Omega_j$ . In other words, for each of  $j = 1, 2$ ,  $B_j$  is the operator converting the incoming field into the scattered field.

**Remark 2.36** Due to Remark 2.35, (120) can be interpreted to mean that the total scattered field  $\psi_1 + \psi_2$  is a superposition of five scattered fields

$$\psi_1 + \psi_2 = \sum_{j=1,5} v_j + O(\mu^2), \quad (132)$$

where

$$v_1 = B_1 \cdot \phi_0, \quad (133)$$

$$v_2 = B_2 \cdot \phi_0, \quad (134)$$

$$v_3 = B_2 \cdot B_1 \cdot \phi_0, \quad (135)$$

$$v_4 = B_1 \cdot B_2 \cdot \phi_0, \quad (136)$$

$$v_5 = B_2 \cdot B_1 \cdot B_2 \cdot \phi_0. \quad (137)$$

Each of the five scattered fields is generated in a scattering process described as follows:

1. The incoming field  $\phi_0$  gets scattered by the scatterer in  $\Omega_1$ , generating the scattered field  $v_1$ . We denote this scattering process schematically by the chart

$$\phi_0 \longrightarrow \Omega_1 \longrightarrow v_1; \quad (138)$$

2. The incoming field  $\phi_0$  gets scattered by the scatterer in  $\Omega_2$ , generating a scattered field  $v_2$ . We denote this process by the chart

$$\phi_0 \longrightarrow \Omega_2 \longrightarrow v_2; \quad (139)$$

3. The scattered field  $v_1$  enters the scatterer in  $\Omega_2$  as an incoming field, generating the scattered field  $v_4$ . We denote this process by the chart

$$\phi_0 \longrightarrow \Omega_1 \longrightarrow v_1 \longrightarrow \Omega_2 \longrightarrow v_4; \quad (140)$$

4. The scattered field  $v_2$  enters the scatterer in  $\Omega_1$  as an incoming field, and is again scattered by  $\Omega_1$ , generating a scattered field  $v_3$ . This process is denoted by

$$\phi_0 \longrightarrow \Omega_2 \longrightarrow v_2 \longrightarrow \Omega_1 \longrightarrow v_3; \quad (141)$$

5. The scattered field  $v_3$  enters the scatterer in  $\Omega_2$  as an incoming field, generating the scattered field  $v_5$ . This process is denoted by the chart

$$\phi_0 \longrightarrow \Omega_2 \longrightarrow v_2 \longrightarrow \Omega_1 \longrightarrow v_4 \longrightarrow \Omega_2 \longrightarrow v_5. \quad (142)$$

The following lemma is a restatement of Lemma 2.34 in the special case when  $\Omega_1$  is the annulus  $A(R, h)$ , and  $\Omega_2$  is the disk  $D(R)$ .

**Lemma 2.37** *Suppose that  $R \geq 0$ ,  $h > 0$  are two real numbers and that  $k \in C^+$ . Suppose further that  $\phi_0 : D(R+h) \mapsto C$  is an incoming field to the disk  $D(R+h)$ ; in other words, there exists  $\alpha \in Y_{k(R+h)}$ , such that*

$$\phi_0(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot \alpha \quad (143)$$

*inside the disk  $D(R+h)$ . Finally, suppose that  $\psi : R^2 \mapsto C$  is the corresponding scattered field from  $D(R+h)$ . Then for sufficiently small  $h > 0$  and  $r > R+h$*

$$\psi(r, \theta) = \sum_{j=1,5} v_j(r, \theta) + O(h^2), \quad (144)$$

where

$$v_1(r, \theta) = F_\theta^{-1} \cdot H_{kr} \cdot S_{\beta\alpha} \cdot \alpha, \quad r > R+h, \quad (145)$$

$$v_2(r, \theta) = F_\theta^{-1} \cdot H_{kr} \cdot S_{R,k}^- \cdot \alpha, \quad r > R, \quad (146)$$

$$v_3(r, \theta) = F_\theta^{-1} \cdot H_{kr} \cdot S_{R,k}^- \cdot S_{\alpha\alpha} \cdot \alpha, \quad r > R, \quad (147)$$

$$v_4(r, \theta) = F_\theta^{-1} \cdot H_{kr} \cdot S_{\beta\beta} \cdot S_{R,k}^- \cdot \alpha, \quad r > R+h, \quad (148)$$

$$v_5(r, \theta) = F_\theta^{-1} \cdot H_{kr} \cdot S_{R,k}^- \cdot S_{\alpha\beta} \cdot S_{R,k}^- \cdot \alpha, \quad r > R. \quad (149)$$

The following lemma is analogous to Lemma 2.37. it is a restatement of Lemma 2.34 in the special case when  $\Omega_1$  is the annulus  $A(R, h)$ , and  $\Omega_2$  is the disk  $E(R)$ .

**Lemma 2.38** *Suppose that  $R, h$  are two positive numbers and that  $k \in C^+$ . Suppose further that  $\phi_0 : E(R) \mapsto C$  is an incoming field to the scatterer  $q_{E(R)}$ , and that  $\psi : D(R) \mapsto C$  is the corresponding scattered field. Then*

$$\psi(r, \theta) = \sum_{j=1}^5 \psi_j(r, \theta) + O(h^2), \quad r < R, \quad (150)$$

with

$$\psi_1(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot S_{\alpha\beta} \cdot \beta, \quad r < R, \quad (151)$$

$$\psi_2(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot S_{R+h,k}^+ \cdot \beta, \quad r < R+h, \quad (152)$$

$$\psi_3(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot S_{R+h,k}^+ \cdot S_{\beta\beta} \cdot \beta, \quad r < R+h, \quad (153)$$

$$\psi_4(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot S_{\alpha\alpha} \cdot S_{R+h,k}^+ \cdot \beta, \quad r < R, \quad (154)$$

$$\psi_5(r, \theta) = F_\theta^{-1} \cdot J_{kr} \cdot S_{R+h,k}^+ \cdot S_{\beta\alpha} \cdot S_{R+h,k}^+ \cdot \beta, \quad r < R+h. \quad (155)$$



**Remark 2.39** The five scattered fields  $\psi_j$ ,  $j = 1, 5$  in (150)–(155) are generated in the scattering process described as follows:

$$1) \quad \phi_0 \rightarrow A(R, h) \rightarrow \psi_1, \quad (156)$$

$$2) \quad \phi_0 \rightarrow E(R + h) \rightarrow \psi_2, \quad (157)$$

$$3) \quad \phi_0 \rightarrow A(R, h) \rightarrow \psi_1 \rightarrow E(R + h) \rightarrow \psi_3, \quad (158)$$

$$4) \quad \phi_0 \rightarrow E(R + h) \rightarrow \psi_2 \rightarrow A(R, h) \rightarrow \psi_4, \quad (159)$$

$$5) \quad \phi_0 \rightarrow E(R + h) \rightarrow \psi_2 \rightarrow A(R, h) \rightarrow \psi_4 \rightarrow E(R + h) \rightarrow \psi_5. \quad (160)$$

### 3 Riccati Equations for Scattering Matrices

It turns out that the scattering matrices  $S_{r,k}^-$  and  $S_{r,k}^+$ , viewed as functions of  $r$ , satisfy certain matrix Riccati equations with respect to  $r$ . In this section, we derive these Riccati equations.

#### 3.1 Scattering Matrices for Thin Annuli

In this subsection, we obtain approximate expressions for the four submatrices  $S_{\beta\alpha}$ ,  $S_{\beta\beta}$ ,  $S_{\alpha\alpha}$  and  $S_{\alpha\beta}$  of the matrix (98) when the annulus  $A(R, h)$  is thin (i.e.,  $h$  is small). In this case, the scattering from  $q_{A(R,h)}$  is weak, and  $S_{\beta\alpha}$ ,  $S_{\beta\beta}$ ,  $S_{\alpha\alpha}$  and  $S_{\alpha\beta}$  assume a particularly simple form.

**Lemma 3.1** (*Born Approximation*) Suppose that  $R \geq 0$ ,  $h > 0$  are two real numbers and that  $k \in C^+$ . Suppose further that  $\phi_0 : A(R, h) \mapsto C$  is an incoming field to the annular scatterer  $q_{A(R,h)}$ , and that  $\phi_0^R$  is the restriction of  $\phi_0$  on the circle  $C_R \subset R^2$ . Finally, suppose that  $\psi : R^2 \mapsto C$  is the scattered field. Then for small  $h$ ,

$$\psi(r, \theta) = \frac{i \cdot h \cdot k^2 \cdot \pi \cdot R}{2} F_\theta^{-1} \cdot J_{kr} \cdot H_{kr} \cdot F \cdot Q_R \cdot \phi_0^R + O(h^2) \quad (161)$$

for all  $r \leq R$ , and

$$\psi(r, \theta) = \frac{i \cdot h \cdot k^2 \cdot \pi \cdot R}{2} F_\theta^{-1} \cdot J_{kr} \cdot H_{kr} \cdot F \cdot Q_R \cdot \phi_0^R + O(h^2) \quad (162)$$

for all  $r \geq R + h$ .

**Proof.** The scattered field  $\psi$  satisfies the Lippmann-Schwinger equation (see Remark 2.17)

$$\psi(x) + k^2 \int_{A(R,h)} G_k(x, \xi) q(\xi) \psi(\xi) d\xi = -k^2 \int_{A(R,h)} G_k(x, \xi) q(\xi) \phi_0(\xi) d\xi \quad (163)$$

whose solution can be approximated by the sequence  $\{ \psi_m, m = 1, 2, \dots \}$  generated by the fixed-point iteration (72), (73) so that

$$\psi_1(x) = k^2 \int_{A(R,h)} G_k(x, \xi) q(\xi) \phi_0(\xi) d\xi \quad (164)$$

$$\psi_{m+1} = g - G_k^q \psi_m, \quad m = 1, 2, \dots; \quad (165)$$

usually,  $\psi_1$  is referred to as the Born approximation to the scattered field  $\psi$ . According to Lemma 2.19, there exist positive numbers  $\epsilon > 0$ ,  $c > 0$  such that

$$| \psi(x) - \psi_1(x) | \leq c \cdot h^2 \quad (166)$$

for all  $h < \epsilon$ ,  $x \in R^2$ . The combination of (164), (48) and (51) yields

$$\begin{aligned} \psi_1(r, \theta) = & \frac{i}{4} k^2 \int_R^{R+h} \int_0^{2\pi} q(r', \theta') \phi_0(r', \theta') \times \\ & \sum_{m=-\infty}^{\infty} J_m(kr) H_m(kr') e^{im(\theta-\theta')} r' dr' d\theta', \end{aligned} \quad (167)$$

for all  $r \leq R$ , and

$$\begin{aligned} \psi_1(r, \theta) = & \frac{i}{4} k^2 \int_R^{R+h} \int_0^{2\pi} q(r', \theta') \phi_0(r', \theta') \times \\ & \sum_{m=-\infty}^{\infty} J_m(kr') H_m(kr) e^{im(\theta-\theta')} r' dr' d\theta', \end{aligned} \quad (168)$$

for all  $r \geq R + h$ . Obviously, for any function  $g \in C^1[0, \infty)$ ,

$$\int_R^{R+h} g(t) dt = h \cdot (g(R) + O(h)), \quad (169)$$

and combining (167), (168) with (169), we obtain

$$\begin{aligned} \psi_1(r, \theta) = & \frac{i}{4} k^2 h R \sum_{m=-\infty}^{\infty} \left( J_m(kr) H_m(kR) \int_0^{2\pi} e^{-im\theta'} q(R, \theta') \phi_0(R, \theta') d\theta' \right) e^{im\theta} \\ & + O(h^2). \end{aligned} \quad (170)$$

for all  $r \leq R$ , and

$$\begin{aligned} \psi_1(r, \theta) = & \frac{i}{4} k^2 h R \sum_{m=-\infty}^{\infty} \left( J_m(kR) H_m(kr) \int_0^{2\pi} e^{-im\theta'} q(R, \theta') \phi_0(R, \theta') d\theta' \right) e^{im\theta} \\ & + O(h^2). \end{aligned} \quad (171)$$

for all  $r \geq R + h$ . Using the notation introduced in Section 2.1, (170), (171) can be rewritten in the form

$$\psi_1(r, \theta) = h \frac{i}{4} k^2 2\pi R \cdot F_\theta^{-1} \cdot J_{kr} \cdot H_{kR} \cdot F \cdot Q_R \cdot \phi_0^R + O(h^2), \quad (172)$$

for all  $r \leq R$ , and

$$\psi_1(r, \theta) = h \frac{i}{4} k^2 2\pi R \cdot F_\theta^{-1} \cdot J_{kR} \cdot H_{kr} \cdot F \cdot Q_R \cdot \phi_0^R + O(h^2), \quad (173)$$

for all  $r \geq R + h$ . Now, the lemma follows from the combination of (166), (172) and (173).  $\square$

The following lemma provides the desired approximate expressions for the scattering matrix  $S_{R,k}^h$ . It is a direct result of formulae (161) and (162).

**Lemma 3.2** *Suppose that under the conditions of Lemma 3.1, the four submatrices of the scattering matrix  $S_{\beta\alpha}$ ,  $S_{\beta\beta}$ ,  $S_{\alpha\alpha}$  and  $S_{\alpha\beta}$  defined by the formulae (97), (98)). Then*

$$S_{\beta\alpha} = h \frac{i\pi}{2} k^2 R \cdot J_{kR} \cdot F \cdot Q_R \cdot F^{-1} \cdot J_{kR} + O(h^2), \quad (174)$$

$$S_{\beta\beta} = h \frac{i\pi}{2} k^2 R \cdot J_{kR} \cdot F \cdot Q_R \cdot F^{-1} \cdot H_{kR} + O(h^2), \quad (175)$$

$$S_{\alpha\alpha} = h \frac{i\pi}{2} k^2 R \cdot H_{kR} \cdot F \cdot Q_R \cdot F^{-1} \cdot J_{kR} + O(h^2), \quad (176)$$

$$S_{\alpha\beta} = h \frac{i\pi}{2} k^2 R \cdot H_{kR} \cdot F \cdot Q_R \cdot F^{-1} \cdot H_{kR} + O(h^2). \quad (177)$$

**Proof.** We will prove only formula (174), the proofs for the rest being similar. According to Lemma 2.27, an incoming field  $\phi_0$  to  $A(R, h)$  has the form

$$\phi_0 = F_\theta^{-1} \{J_{kr}\alpha + H_{kr}\beta\}, \quad (178)$$

with  $\alpha \in Y_{k(R+h)}$ ,  $\beta \in X_{kR}$ . Setting  $\beta = 0$ , and substituting (178) into (161), we obtain the scattered field  $\psi$  outside  $D(R + h)$  in the form

$$\psi(r, \theta) = F_\theta^{-1} H_{kr} \tilde{\beta} = h \frac{i\pi}{2} k^2 R \cdot F_\theta^{-1} H_{kr} J_{kR} F Q_R F^{-1} J_{kR} \alpha + O(h^2), \quad (179)$$

that is,

$$\tilde{\beta} = h \frac{i\pi}{2} k^2 R \cdot J_{kR} F Q_R F^{-1} J_{kR} \alpha + O(h^2). \quad (180)$$

On the other hand, since  $\beta = 0$ , (99) assumes the form

$$\tilde{\beta} = S_{\beta\alpha} \alpha, \quad (181)$$

and (174) follows from the combination of (180), (181).  $\square$

### 3.2 A Riccati Equation for the Scattering Matrix $S^-$

In this subsection, we demonstrate that the scattering matrix  $S_{r,k}^-$  as a function of  $r$  is a solution of a Riccati equation. To this end, we calculate the difference  $S_{R+h,k}^- - S_{R,k}^-$ , to second order in  $h$ , by observing that the scattered field from the scatterer  $q_{D(R+h)}$  is the result of combined effect from the two scatterers  $q_{D(R)}$  and  $q_{A(R,h)}$ .

According to Lemma 2.20, an incoming field inside the disk  $D(R+h)$  has the form

$$\phi_0 = F_\theta^{-1} J_{kr} \alpha, \quad (182)$$

with  $\alpha \in Y_{k(R+h)}$ . The corresponding scattered field outside the disk  $D(R+h)$  has the form

$$\psi = F_\theta^{-1} H_{kr} \beta, \quad (183)$$

with  $\beta \in X_{k(R+h)}$ . By the definition of the scattering matrix  $S_{R+h,k}^-$  (see (87), (83))

$$\beta = S_{R+h,k}^- \alpha. \quad (184)$$

The following result is an immediate consequence of Lemmas 2.37 and 3.2.

**Lemma 3.3** *Suppose that  $R \geq 0$ ,  $h > 0$  are two real numbers, that  $\phi_0 : D(R+h) \mapsto C$  is an incoming field to the disk scatterer in  $D(R+h)$ , and that  $\psi : R^2 \mapsto C$  is the scattered field. Then*

$$\beta = \left\{ S_{R,k}^- + h \frac{i\pi R}{2} k^2 (J_{kR} + S_{R,k}^- H_{kR}) F Q_R F^{-1} (H_{kR} S_{R,k}^- + J_{kR}) \right\} \alpha + O(h^2).$$

The following theorem is an immediate consequence of (184) and Lemma 3.3.

**Theorem 3.4** *(Riccati Equation for the Scattering Matrix  $S^-$ ) For any  $k \in C^+$  and all  $r \geq 0$ , the scattering matrix  $S_{r,k}^- : Y_{kr} \mapsto X_{kr}$  is a solution of the Riccati equation*

$$\frac{dS_{r,k}^-}{dr} = \frac{i\pi r}{2} k^2 (J_{kr} + S_{r,k}^- H_{kr}) F Q_r F^{-1} (H_{kr} S_{r,k}^- + J_{kr}). \quad (185)$$

### 3.3 A Riccati Equation for the Scattering Matrix $S^+$

In this subsection, we derive a Riccati equation for the exterior scattering matrix  $S_{r,k}^+$  defined in (90). We will only state the results, since their proofs are quite similar to those for the interior scattering matrix in the preceding subsection.

According to Lemma 2.21, to  $E(R)$  (the exterior of the disk  $D(R)$ ), an incoming field assumes the form

$$\phi_0(r, \theta) = F_\theta^{-1} H_{kr} \beta, \quad (186)$$

with  $\beta \in X_{kR}$ . The corresponding scattered field inside the disk  $D(R)$  assumes the form

$$\psi(r, \theta) = F_\theta^{-1} J_{kr} \alpha, \quad (187)$$

with  $\alpha \in Y_{kR}$ . By the definition of  $S_{R,k}^+$  (see (90)),

$$\alpha = S_{R,k}^+ \beta. \quad (188)$$

Since

$$E(R) = E(R+h) \cup A(R, h), \quad (189)$$

the field  $\psi$  scattered by the scatterer  $q_{E(R)}$  can be viewed as the field scattered by the combination of scatterers in  $A(R, h)$  and  $E(R+h)$ . The following lemma is analogous to Lemma 2.37, and is an immediate consequence of Lemmas 2.38 and 3.2.

**Lemma 3.5** *Suppose that  $R, h$  are two positive numbers and that  $k \in C^+$ . Suppose further that  $\phi_0 : E(R) \mapsto C$  is an incoming field (186) to the scatterer  $q_{E(R)}$ , and that  $\psi : D(R) \mapsto C$  is the corresponding scattered field (187). Then*

$$\begin{aligned} \beta = & \left\{ S_{R+h,k}^+ + h \frac{i\pi(R+h)}{2} k^2 (H_{k(R+h)} + S_{R+h,k}^+ J_{k(R+h)}) F Q_{R+h} F^{-1} \times \right. \\ & \left. (J_{k(R+h)} S_{R+h,k}^+ + H_{k(R+h)}) \right\} \alpha + O(h^2). \end{aligned} \quad (190)$$

The following theorem is an immediate consequence of (188) and Lemma 3.5.

**Theorem 3.6** *(Riccati Equation for the Scattering Matrix  $S^+$ ) For any  $k \in C^+$  and all  $r > 0$ , the exterior scattering matrix  $S_{r,k}^+ : X_{kr} \mapsto Y_{kr}$  is a solution of the Riccati equation*

$$\frac{dS_{r,k}^+}{dr} = -\frac{i\pi r}{2} k^2 (H_{kr} + S_{r,k}^+ J_{kr}) F Q_r F^{-1} (J_{kr} S_{r,k}^+ + H_{kr}). \quad (191)$$

## References

- [1] Eyges, L. (1972), *The Classical Eletromagnetic Field*, Dover, New York.
- [2] K. Chadan and P.C. Sabatier (1977), *Inverse problems in quantum scattering theory*, Springer-Verlag.
- [3] M. Kaveh and M. Soumekh (1987), *Computer-assisted diffraction tomography*, in *Image Recovery: Theory and Application*, Henry Stark, Ed. Academic Press, Inc., pp. 369–413.
- [4] W. W. Symes and J. J. Carazzone (1989), *Velocity inversion by coherency optimization*, to appear in *Geophysics*.
- [5] L. Lines and S. Treitel (1984), *A review of least-squares inversion and its applications to geophysical problems*, *Geophysical Prospecting* 32, pp. 159–186.
- [6] G. S. Pan and R. A. Phinney (1989), *Full-waveform inversion of plane-wave seismogram in stratified acoustic media: Applicability and limitations*, *Geophysics* 54, pp. 568–580.
- [7] I.M. Gel'fand and B.M. Levitan (1951), *On the determination of a differential equation by its spectral function*, *Dokl. Akad. USSR* 77, pp. 557–560.
- [8] V.A. Marchenko (1955), *Reconstruction of potential energy from the phase of scattered waves*, *Dokl. Akad. USSR* 104, pp. 695–698.
- [9] L. D. Faddeyev (1961), *The inverse problem in the quantum theory of scattering*, *Journal of Mathematical Physics*, vol. 4, No. 1, 72–104.
- [10] P. Deift and E. Trubowitz (1979), *Inverse scattering on the line*, *Comm. Pure Appl. Math*, vol. 32, pp. 121–251.
- [11] W.Y. Crutchfield (1983), *Class of exact inversion solutions to vibrating string problem*, *Physics Letters*, vol. 98A, Number 5,6, pp. 233–236.
- [12] D.C. Stickler (1984), *Application of the trace formula methods to inverse scattering for some geophysical problems*, *SIAM AMS proceedings* vol. 14, *Inverse Problems*, pp 13–30
- [13] J. Sylvester and G. Uhlmann (1986), *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, *Comm. Pure Appl. Math.* 17, pp. 91–112.

- [14] J. Somersalo, M. Cheney, D. Isaacson and E. L. Isaacson (1991), *Layer stripping: A Direct numerical method for impedance imaging*, *Inverse Problems* 7 899–926.
- [15] R. G. Newton (1989), *Inverse Schrödinger Scattering in Three Dimensions*, Springer-Verlag
- [16] Y. Chen and V. Rokhlin (1992), *On the inverse scattering problem for the Helmholtz equation in one dimension*, *Inverse Problems* 8 356–391.
- [17] J. Sylvester (1992), *A Convergent Layer Stripping Algorithm for the Radially Symmetric Impedance Tomography Problem*, *Communications in PDE* 17, No.12, pp.1955–1994.
- [18] Ikebe, T. (1959), *Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory*, *Arch. Rational Mech. Anal.* Vol. 5. 1960, pp. 1–34
- [19] Courant, R., and Hilbert, D. (1962), *Methods of Mathematical Physics*, Vol. 2. Wiley (Interscience), New York.
- [20] D. Colton and R. Kress (1983), *Inverse Acoustic and Electromagnetic Scattering Theory*, Wiley-Interscience Publication, New York.
- [21] Abramowitz, M., and Stegun, I. (1965), *Handbook of Mathematical Functions*, Dover, New York.